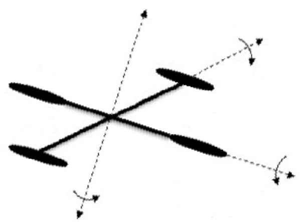
3-4. 미정계수법



Example 1 General Solution Using Undetermined Coefficients

$$y'' + 4y' - 2y = 2x^2 - 3x + 6.$$

Solution

Step 1: $y'' + 4y' - 2y = 0 \;\rightarrow\; m^2 + 4m - 2 = 0 \;\rightarrow\; m_{1,2} = -2 \pm \sqrt{6}$

$$y_c = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x}.$$

Step 2: $y_p = Ax^2 + Bx + C. \;\rightarrow\; y_p' = 2Ax + B, \quad y_p'' = 2A$

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 2x^2 - 3x + 6.$$

$$-2Ax^2 + (8A - 2B)x + (2A + 4B - 2C) = 2x^2 - 3x + 6$$

$$-2A = 2, \quad 8A - 2B = -3, \quad 2A + 4B - 2C = 6.$$

$$A = -1, \; B = -5/2, \; C = -9$$

$$y_p = -x^2 - \frac{5}{2}x - 9.$$

Step 3: $y = y_c + y_p = c_1 e^{-(2+\sqrt{6})x} + c_2 e^{(-2+\sqrt{6})x} - x^2 - \dfrac{5}{2}x - 9.$

Example 2 Particular Solution Using Undetermined Coefficients

$$y'' - y' + y = 2\sin 3x$$

Solution

$$y_p = A\cos 3x + B\sin 3x.$$

$$y_p'' - y_p' + y_p = (-8A - 3B)\cos 3x + (3A - 8B)\sin 3x = 2\sin 3x$$

$$-8A - 3B = 0, \qquad 3A - 8B = 2,$$

$$y_p = \frac{6}{73}\cos 3x - \frac{16}{73}\sin 3x.$$

Example 3 Forming y_p by Superposition

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}.$$

Solution

Step 1: $y'' - 2y' - 3y = 0 \;\rightarrow\; y_c = c_1 e^{-x} + c_2 e^{3x}$

Step 2: $g(x) = g_1(x) + g_2(x) = \text{polynomial} + \text{exponentials.}$

$$y_p = y_{p_1} + y_{p_2}$$

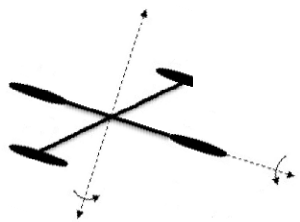
$$y_p = Ax + B + Cxe^{2x} + Ee^{2x}$$

$$y_p'' - 2y_p' - 3y_p = -3Ax - 2A - 3B - 3Cxe^{2x} + (2C - 3E)e^{2x} = 4x - 5 + 6xe^{2x}.$$

$$-3A = 4, \quad -2A - 3B = -5, \quad -3C = 6, \quad 2C - 3E = 0.$$

$$A = -4/3,\; B = 23/9,\; C = -2,\; E = -4/3,$$

$$y_p = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - \frac{4}{3}e^{2x}.$$



Step 3: $y = c_1 e^{-x} + c_2 e^{3x} - \frac{4}{3} + \frac{23}{9} - \left(2x + \frac{4}{3}\right) e^{2x}.$

(Note) $y_{p_1} = Ax + B$ into $y'' - 2y' - 3y = 4x - 5$

$y_{p_2} = Cxe^{2x} + Ee^{2x}$ into $y'' - 2y' - 3y = 6xe^{2x}$

Example 4 A Glitch in the Method

$$y'' - 5y' + 4y = 8e^x$$

Solution

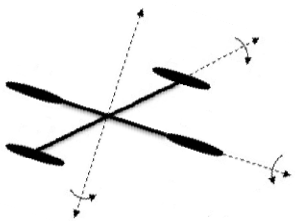
Step 1: $y'' - 5y' + 4y = 0 \rightarrow y_c = c_1 e^x + c_2 e^{4x}$

Step 2: $0 = 8e^x \rightarrow$ Wrong guess for y_p

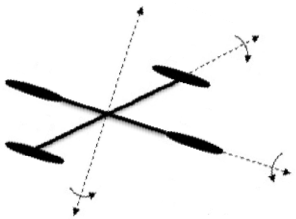
$$y_p = Axe^x.$$

$$y_p'' - 5y_p' + 4y_p = -3Ae^x = 8e^x. \rightarrow A = -8/3$$

$$y_p = -(8/3)xe^x$$



$g(x)$	Form of y_p
1	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
x^3-x+1	Ax^3+Bx^2+Cx+E
$\sin 4x$	$A\cos 4x+B\sin 4x$
$\cos 4x$	$A\cos 4x+B\sin 4x$
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
x^2e^{5x}	$(Ax^2+Bx+C)e^{5x}$
$e^{3x}\sin 4x$	$Ae^{3x}\cos 4x+Be^{3x}\sin 4x$
$5x^2\sin 4x$	$(Ax^2+Bx+C)\cos 4x+(Ex^2+Fx+G)\sin 4x$
$xe^{3x}\cos 4x$	$(Ax+B)e^{3x}\cos 4x+(Cx+E)e^{3x}\sin 4x$



Example 5 Forms of Particular Solutions – Case I

$$(a)\ \ y'' - 8y' + 25y = 5x^2 e^{-x} - 7e^{-x} \qquad (b)\ \ y'' + 4y = x\cos x.$$

Solution

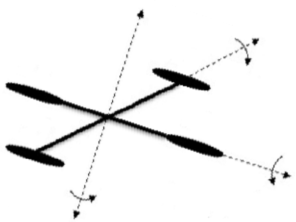
$$(a)\ \ g(x) = (5x^2 - 7)e^{-x}$$

$$y_p = (Ax^3 + Bx^2 + Cx + E)e^{-x}.$$

$$(b)\ \ y_p = (Ax + B)\cos x + (Cx + E)\sin x.$$

Check: y_p 와 $y_c = c_1 \cos 2x + c_2 \sin 2x$

$$y_p = y_{p_1} + y_{p_2} + \cdots + y_{p_m}.$$



Example 6 Forming y_p by Superposition – Case I

$$y'' - 9y' + 14y = 3x^2 - 5\sin 2x + 7xe^{6x}.$$

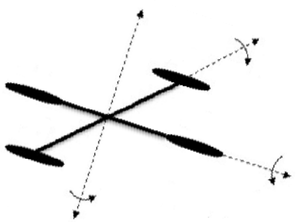
Solution

$$3x^2 \quad \rightarrow \quad y_{p_1} = Ax^2 + Bx + C.$$

$$-5\sin 2x \quad \rightarrow \quad y_{p_2} = E\cos 2x + F\sin 2x.$$

$$7xe^{6x} \quad \rightarrow \quad y_{p_3} = (Gx + H)e^{6x}.$$

$$y_p = y_{p_1} + y_{p_2} + y_{p_3} = Ax^2 + Bx + C + E\cos 2x + F\sin 2x + (Gx + H)e^{6x}.$$



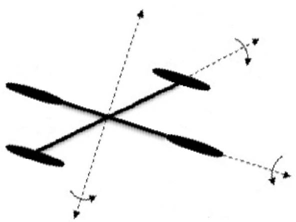
Example 7 Particular Solution – Case II

$$y'' - 2y' + y = e^x.$$

Solution

$$y_p = Ax^2e^x. \qquad 2Ae^x = e^x \rightarrow A = 1/2$$

$$\rightarrow y_p = (1/2)x^2e^x$$



Example 8 **An Initial-Value Problem**

$$y'' + y = 4x + 10\sin x, \quad y(\pi) = 0, \quad y'(\pi) = 2.$$

Solution

$$y_p = Ax + B + C\cos x + E\sin x.$$

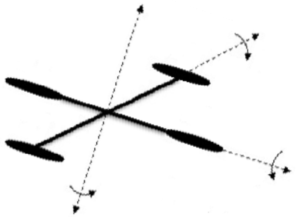
$$y_p = Ax + B + Cx\cos x + Ex\sin x.$$

$$y_p'' + y_p = Ax + B - 2C\sin x + 2E\cos x = 4x + 10\sin x,$$

$$\rightarrow A = 4,\ B = 0,\ C = -5,\ E = 0 \rightarrow y = y_c + y_p = c_1\cos x + c_2\sin x + 4x - 5x\cos x.$$

$$y(\pi) = 0, \quad y'(\pi) = 2. \rightarrow c_1 = 9\pi,\ c_2 = 7$$

$$y = 9\pi\cos x + 7\sin x + 4x - 5x\cos x.$$



Example 9 Using the Multiplication Rule

$$y'' - 6y' + 9y = 6x^2 + 2 - 12e^{3x}$$

Solution

$$y'' - 6y' + 9y = 0 \;\rightarrow\; y_c = c_1 e^{3x} + c_2 x e^{3x}$$

$$g(x) = 6x^2 + 2 - 12e^{3x}$$

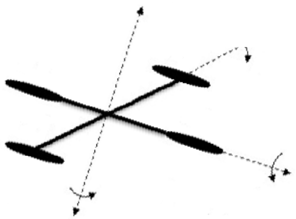
$$y_p = \underbrace{Ax^2 + Bx + C}_{y_{p_1}} + \underbrace{Ee^{3x}}_{y_{p_2}}.$$

$$y_p = Ax^2 + Bx + C + Ex^2 e^{3x}.$$

$$y_p'' - 6y_p' + 9y_p = 9Ax^2 + (-12A + 9B)x + 2A - 6B + 9C + 2Ee^{3x} = 6x^2 + 2 - 12e^{3x}.$$

$$\rightarrow\; A = 2/3,\;\; B = 8/9,\;\; C = 2/3,\;\; E = -6$$

$$\rightarrow\; y = c_1 e^{3x} + c_2 x e^{3x} + \frac{2}{3}x^2 + \frac{8}{9}x + \frac{2}{3} - 6x^2 e^{3x}.$$



Example 10 Third-Order DE – Case I

$$y''' + y'' = e^x \cos x$$

Solution

$y''' + y'' = 0 \;\rightarrow\; m^3 + m^2 = 0 \;\rightarrow\; m = 0$ (roots), $m = -1 \;\rightarrow\; y_c = c_1 + c_2 x + c_3 e^{-x}$

$$y_p = Ae^x \cos x + Be^x \sin x.$$

$$y_p''' + y_p'' = (-2A + 4B)e^x \cos x + (-4A - 2B)e^x \sin x = e^x \cos x$$

$$\rightarrow\; A = -1/10,\;\; B = 1/5$$

$$\rightarrow\; y = y_c + y_p = c_1 + c_2 x + c_3 e^{-x} - \frac{1}{10} e^x \cos x + \frac{1}{5} e^x \sin x.$$

Example 11 Fourth-Order De – Case II

$$y^{(4)} + y''' = 1 - x^2 e^{-x}$$

Solution

$$y^{(4)} + y''' = 0 \rightarrow m^4 + m^3 = 0 \rightarrow m = 1,\ m = -1 \rightarrow y_c = c_1 + c_2 x + c_3 x^2 + c_4 e^{-x}$$

$$y_p = \underbrace{A}_{y_{p_1}} + \underbrace{Bx^2 e^{-x} + Cxe^{-x} + Ee^{-x}}_{y_{p_2}}.$$

$$y_{p1} \times x^3 \rightarrow y_{p1}$$

$$y_{p2} \times x \rightarrow y_{p2}$$

$$y_p = Ax^3 + Bx^3 e^{-x} + Cx^2 e^{-x} + Exe^{-x}.$$